Solution of the sample entrance exam test for FEE CTU

1. The set of all solutions of the equation $2^{|x+3|} < 2$ with the unknown $x \in \mathbb{R}$ is…

$$\begin{aligned} 2^{|x+3|} &< 2 \\ 2^{|x+3|} &< 2^1 \\ |x+3| &< 1 \end{aligned}$$

$$\begin{array}{c|c} x \in (-\infty,\ -3] & x \in (-3,\ \infty) \\ \hline -(x+3) < 1 & (x+3) < 1 \\ x > -4 & x < -2 \\ \hline x \in (-4, -3] & x \in (-3, -2) \end{array}$$

Answer: $x \in (-4, -2)$.

2. The domain of the function $f(x) = \dfrac{x}{\sin x}$ is…

$$\begin{aligned} \sin x &\neq 0 \\ x &\neq k\pi, \quad k \in \mathbb{Z} \end{aligned}$$

Domain: $\bigcup_{k\in\mathbb{Z}}(k\pi,\ (k+1)\pi),\ k \in \mathbb{Z}$.

The correct choice is “the same as the domain of $g(x) = \cot x$”.

3. Assuming that $\sin\alpha \cdot \cos\alpha = \frac{1}{2}$ and $\alpha \in (\pi,\ 2\pi)$, the value of $\tan(\pi - \alpha)$ is…

$$\begin{aligned} \sin\alpha \cdot \cos\alpha &= \tfrac{1}{2} \\ 2\sin\alpha \cdot \cos\alpha &= 1 \\ \sin 2\alpha &= 1 \\ 2\alpha &= \tfrac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z} \\ \alpha &= \tfrac{\pi}{4} + k\pi,\ k \in \mathbb{Z} \\ \alpha \in (\pi,\ 2\pi) \Rightarrow \quad \alpha &= \tfrac{5\pi}{4} \end{aligned}$$

$\tan(\pi - \frac{5\pi}{4}) = \tan(-\frac{\pi}{4}) = -1$.
Answer: $\tan(\pi - \alpha) = -1$.

4. In the interval $[0,\ 2\pi]$, the equation $\sin x = \cos x - 1$ has (how many solution)…

Knowing values of trignometric functions for popular angles, we right away see that equality occurs when $x = 0$ or $x = 2\pi$. We also notice that $\cos\ x - 1 < 0$ on the interval $(0, 2\pi)$, so other possible solutions can be found only in the interval $(\pi, 2\pi)$ where $\sin\ x$ is negative. On this interval we have $\sin\ x = -\sqrt{1 - \cos^2\ x}$.

$$\begin{aligned} \sin\ x &= \cos\ x - 1 \\ -\sqrt{1-\cos^2\ x} &= \cos\ x - 1 \\ 1 - \cos\ x &= \sqrt{1 - \cos^2\ x} \qquad /^2\ LHS > 0,\ RHS > 0 \\ 1 - 2\cos\ x + \cos^2\ x &= 1 - \cos^2\ x \\ 2\cos^2\ x - 2\cos\ x &= 0 \\ \cos\ x(\cos\ x - 1) &= 0 \qquad /:(\cos\ x - 1) \neq 0 \\ \cos\ x &= 0 \\ x &= \tfrac{3\pi}{2} \end{aligned}$$

Answer: The equation has exactly three solutions in the interval $[0,\ 2\pi]$, namely $x = 0, \frac{3\pi}{2}, 2\pi$.

5. The algebraic form of the complex number $z = \dfrac{1+i}{1+2i}$ is...

$$\frac{1+i}{1+2i} = \frac{1+i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{1-2i+i+2}{1+4} = \frac{3-i}{5}$$

Answer: $z = \dfrac{3}{5} - \dfrac{1}{5}i$.

6. The number y determined by the equation $\log_2 y = 3\log_2 \dfrac{x-2}{2} - 2\log_2 \dfrac{x^2-4}{2}$ is equal to...

$$\begin{aligned}
\log_2 y &= 3\log_2 \frac{x-2}{2} - 2\log_2 \frac{x^2-4}{2} \\
\log_2 y &= \log_2 \frac{\left(\frac{x-2}{2}\right)^3}{\left(\frac{x^2-4}{2}\right)^2} \\
y &= \frac{\left(\frac{x-2}{2}\right)^3}{\left(\frac{x^2-4}{2}\right)^2} = \frac{(x-2)^3}{8} \cdot \frac{4}{(x-2)^2(x+2)^2} = \frac{x-2}{2(x+2)^2}
\end{aligned}$$

Answer: $y = \dfrac{x-2}{2(x+2)^2}$.

7. Consider two recurrent sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ given by the following formulas: $a_1 = 3$, $b_1 = 0$, and $a_n = 2 \cdot a_{n-1}$, $b_n = b_{n-1} + a_n$ for $n \geq 2$. Determine b_{11}.

$(a_n)_{n=1}^{\infty}$ is obviously a geometric sequence with common ratio $r = 2$. Thus we can express its n-th term as $a_n = a_1 \cdot r^{n-1} = 3 \cdot 2^{n-1}$. Regarding the sequence $(b_n)_{n=1}^{\infty}$, the right approach is to look at the first terms.

$$\begin{aligned}
b_1 &= 0 \\
b_2 &= b_1 + a_2 \\
b_3 &= b_2 + a_3 = b_1 + a_2 + a_3 \\
b_4 &= b_3 + a_4 = b_1 + a_2 + a_3 + a_4 \\
&\vdots \\
b_n &= b_1 + \textstyle\sum_{i=2}^{n} a_i
\end{aligned}$$

Thus $b_{11} = 0 + \sum_{i=2}^{11} 3 \cdot 2^{i-1} = 3 \cdot \sum_{i=1}^{10} \cdot 2^i = 3 \cdot (2^{11} - 2) = 3 \cdot (2048 - 2) = 6138$.

Answer: $b_{11} = 6138$.

8. Four numbers were inserted between numbers 7 and 22 so that these six numbers form the first six successive terms of a certain arithmetic sequence. The sum of its first eight terms is...

$$a_1 = 7,\ a_6 = 22 = a_1 + 5d \ \Rightarrow\ d = 3 \ \Rightarrow\ a_8 = a_1 + 7d = 7 + 7 \cdot 3 = 28$$

$s_8 = a_1 + a_2 + \ldots + a_8 = \dfrac{a_1 + a_8}{2} \cdot 8 = \dfrac{7+28}{2} \cdot 8 = 140$

Answer: $s_8 = 140$.

9. The expression $\dfrac{6x^3b^3}{25y^4} \cdot \dfrac{15y}{b^2}$ simplifies to...

We just need to cancel and take care of conditions for existence.

Answer: $\dfrac{18bx^3}{5y^3}$, assuming that $y \neq 0 \wedge b \neq 0$.

10. The graph of the function $y = \left(\dfrac{1-\sqrt{x}}{\sqrt{x}-x}\right)^2$ is a part of...

$$y = \left(\frac{1-\sqrt{x}}{\sqrt{x}-x}\right)^2 = \left(\frac{1-\sqrt{x}}{\sqrt{x}(1-\sqrt{x})}\right)^2 = \left(\frac{1}{\sqrt{x}}\right)^2 = \frac{1}{x},\ x > 0$$

The graph of the rational function $\frac{1}{x}$ is a hyperbola.
Answer: The graph of the given function is a part of a hyperbola.

11. The set of all solutions of the inequality $|x+5| \geq 4 + |3-2x|$ with the unknown $x \in \mathbb{R}$ is...

$x \in (-\infty, -5]$	$x \in (-5, \frac{3}{2}]$	$x \in (\frac{3}{2}, \infty)$
$-(x+5) \geq 4+(3-2x)$	$(x+5) \geq 4+(3-2x)$	$(x+5) \geq 4-(3-2x)$
$-x-5 \geq 4+3-2x$	$x+5 \geq 4+3-2x$	$x+5 \geq 4-3+2x$
$x \geq 12$	$3x \geq 2$	$-x \geq -4$
	$x \geq \frac{2}{3}$	$x \leq 4$
$\emptyset$	$x \in [\frac{2}{3}, \frac{3}{2}]$	$x \in (\frac{3}{2}, 4]$

Answer: $x \in [\frac{2}{3}, 4]$.

12. Consider the straight lines that pass through the point $A = (0, -5)$ and whose distance from the origin is $\sqrt{5}$. Their slopes are...
We seek straight lines that are tangent to the circle centered at the origin O and with radius $\sqrt{5}$ and that pass through the point A that lies on the y-axis. This means that the situation is symmetric with respect to the y-axis, and thus the slopes will have symmetric values $\pm k$. Therefore it is enough to focus on the line with positive slope. It will intersect the x-axis at some point $B = (b, 0)$. The area of the right-angle triangle AOB can be evaluated using the legs as $\frac{5 \cdot b}{2}$, and also using the hypotenuse and height as $\frac{|AB| \cdot \sqrt{5}}{2}$, where $|AB| = \sqrt{25+b^2}$. These two expressions for the area allow us to determine b:

$$\begin{aligned}
\frac{5 \cdot b}{2} &= \frac{\sqrt{25+b^2} \cdot \sqrt{5}}{2} \\
25b^2 &= (25+b^2) \cdot 5 \\
5b^2 &= 25+b^2 \\
b^2 &= \tfrac{25}{4} \\
b &= \tfrac{5}{2}
\end{aligned}$$

We squared the equation, but since we assume that we work with positive numbers, the solution is valid. The slope of the line passing through the points $A = (0, -5)$ and $B = (\frac{5}{2}, 0)$ is $k = 5 : \frac{5}{2} = 2$. The second point $B' = (-b, 0)$ is symmetric with B and the slope of the line passing through the points A and B' will be $k' = -2$.
Answer: The slopes are -2 and 2.

13. Consider the following sets: $A = \{1, 2, \ldots, 1000\}$, $B = \{x \in A : \dfrac{x}{6} \in \mathbb{Z}\}$, $C = \{x \in A : \dfrac{x}{8} \in \mathbb{Z}\}$, $D = \{x \in A : 237 \leq x \leq 356\}$ (here $\mathbb{Z}$ denotes the set of all integers). How many elements does the set $(B \cap C) \cup D$ have?
The set B consists of multiples of 6, the set C consists of multiples of 8, hence their intersection is the set of all multiples of 24 that are in the set A. There are exactly 41 of them between 1 and 1000. We need to join these with the set D that has exactly 120 elements. How much do they overlap?

$$B \cap C = \{24, 48, \ldots,\ 216, \underline{240}, \underline{264}, \underline{288}, \underline{312}, \underline{336}, 360, \ldots, 984\}$$

$$D = \{237, 238, 239, \underline{240}, 241, \ldots, 355, 356\}$$

The sets $(B \cap C)$ and D have 5 common elements 240, 264, 288, 312 and 336. Their union therefore consists of 41+120-5=156 elements.
Answer: The set $(B \cap C) \cup D$ has 156 elements.

14. The solutions of the equation $x^2 - (p+1)x + 4 = 0$ (with unknown x) are not real exactly if…
A quadratic equation has no real solutions if and only if its discriminant Δ is negative.

$$\Delta = (p+1)^2 - 4 \cdot 4 = p^2 + 2p - 15 = (p+5)(p-3) < 0$$

Answer: $p \in (-5, 3)$.

15. How many characters of the Morse code can be created assuming that characters are created by joining dots and dashes into groups of one, two, three, four or five?
We are creating sequences of length one to five out of two symbols (dot, dash). For a given length we easily determine the number of possibilities, then we add them:

$$2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 2 + 4 + 8 + 16 + 32 = 62.$$

Answer: There are 62 possible characters.

Correct answers in the test: ceddbacabbedeca